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Entropy of Generalized Gaussian Optical Fields

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We consider two definitions of entropy: thermodynamic entropy and signal entropy. We compare their value for the class of generalized Gaussian fields. The first definition is well adapted to monomode stationary fields, while the second one is bounded only for multimode fields. We prove that these two notions are definitely different, for example, the real Gaussian field has a maximum signal entropy and a minimum thermodynamic entropy (among the Gaussian fields).

KEY WORDS: Thermodynamic and signal entropies; generalized Gaussian optical fields; density matrix describing monomode fields; temporal distribution of random electrical field for multimode case.

1. INTRODUCTION

Entropy is an important notion in physics since it measures the degree of formation we can obtain from a system. If entropy grows, our knowledge of the state of the system decreases. We can also say that it measures the disorder.

This paper is concerned with entropy of optical electromagnetic (em) fields. More precisely, we want to calculate the entropy of generalized Gaussian fields which were introduced recently.[1,2]

It is well known that entropy is minimum for a system in a pure state and maximum for a thermal field.[3] What can we say about other fields?

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Different definitions of entropy are encountered in the literature, and we show that they are not identical. Let us recall briefly the two more common definitions of entropy: thermodynamic entropy and signal entropy.

2. TWO DEFINITIONS OF ENTROPY

2.1. Thermodynamic Entropy

For a quantum system defined by its density matrix $\hat{\rho}$ the entropy S is given by the following expression[11]

$$S = -k \operatorname{Tr}\{\hat{\rho} \log \hat{\rho}\} \tag{1}$$

which is independent of the basis since it does not vary through any unitary transformation of basis vectors.

For example, a monomode stationary em field has a density matrix which is diagonal on the n photon states

$$\hat{\rho} = \sum_n \rho_{nn} |n\rangle\langle n| \tag{2}$$

with $\operatorname{Tr} \hat{\rho} = \sum \rho_{nn} = 1$.

It can be shown[3] that when we fix the mean number of photons $\langle n\rangle = \sum n\rho_{nn}$, then (a) $S = 0$ for the field which has exactly n photons, and (b) S is maximum for the chaotic field (or stationary Gaussin field[1,2]), for which

$$\rho_{nn} = \langle n\rangle^n/(1 + \langle n\rangle)^{n+1} \tag{3}$$

If the mean number of photons $\langle n\rangle$ is not determined, the maximum entropy is obtained for the system which is in each of its eigenstates with an equal probability[4]; this situation corresponds to maximum disorder.

The quantum definition of entropy [Eq. (1)] is traced from the thermodynamic entropy of an ensemble of N particles in classical statistics

$$S = -k \int D \log(\Gamma_{2n} D)\, d\Omega \tag{4}$$

where $D = D(x_1, ..., x_n; p_1, ..., p_n)$ is the probability distribution of n particles in the $6n$-dimensional phase space, and Γ_{2n} is a positive quantity so that $\Gamma_{2n} D$ is dimensionless.

The two definitions appearing in Eqs. (1) and (4) are equivalent; this property was recently illustrated by the study of an infinite chain of oscillators. The classical problem was solved by Huerta and Robertson[5]; they obtained the temporal evolution of the Liouville function reduced to N oscillators, and they showed that a system of initially independent oscillators (which are not initially at thermal equilibrium since they are coupled) tends to thermal equilibrium when $t \to \infty$; the entropy becomes then maximum and the Liouville function factorizes.

The same calculation was done quantum mechanically[6] from the density operator $\hat{\rho}$ expressed in terms of operators $\hat{p}$ and $\hat{q}$ by using the Weyl rule which associated to $\hat{\rho}$ a function $f^W(p, q)$ (Wigner function) which plays the same role as the classical Liouville function. The calculation proves that the system thermalizes,

$$\rho \to e^{-H/kT} \qquad \text{with} \qquad H = \sum_i \frac{p_i^2}{2m\Omega} + \frac{m\Omega^2}{2} q_i^2$$

2.2. Signal Entropy

In communication theory, Shannon[7,8] introduced another definition of entropy which is expressed formally as in Eq. (4) but where the indices $1,..., n$ refer to signal samples and not to particles. The distribution D we want to study then becomes the distribution $p(x_1 ,..., x_n ; y_1 ,..., y_n)$ at times $(t_1 ,..., t_n)$ of the random complex signal $Z(t) = x(t) + iy(t)$.

The signal entropy is defined after sampling the em field without any reference to oscillators. The Shannon entropy is thus defined as a measure of information by the relation

$$\begin{aligned} H &= \lim_{n\to\infty} (1/n)\, H_n \\ H_n &= -\int \cdots \int p(x_1 ,..., x_n ; y_1 ,..., y_n) \\ &\qquad \times \log p(x_1 ,..., x_n ; y_1 ,..., y_n) \prod_1^n dx_i\, dy_i \end{aligned} \tag{5}$$

Shannon showed[8] that among the real random functions (r.f.) which have a given set of moments $\langle x(t_i)\, x(t_j)\rangle$, the r.f. that presents the greatest entropy is the Gaussian one. What can we affirm for a complex r.f. $Z(t) = X(t) + iY(t)$? First of all we show that entropy is maximum when $X(t)$ and $Y(t)$ are independent Gaussian r.f.'s (see Appendix A).

When $Z(t)$ is the complex amplitude of an em field $E^+(t)$ $[= Z(t)\, e^{i\omega_0 t}]$ the independence of $X(t)$ and $Y(t)$ is realized (a) for a monomode chaotic field, and (b) for a multimode nonchaotic field such that $X(t)$ and $Y(t)$ are independent Gaussian r.f.'s; we shall this call it an "independent Gaussian field."

We have to notice that Shannon's definition [Eq. (5)] depends on the units chosen. In communication theory this problem has no importance since we have generally to compare real (resp. complex) signals with real (resp. complex) ones. But here we want to compare the entropy of a real r.f. with the entropy of a complex r.f., since the amplitude $Z(t)$ of $E^+(t)$ can be real or complex. Then, as in classical thermodynamics, we must introduce a constant Γ_{2N} such that the product $\Gamma_{2N}\, p(x_1, ..., x_N; y_1, ..., y_N)$ is dimensionless. Moreover, the additivity of entropies of two independent systems necessitates[10] that if **x** and **y** are independent and have the same distribution, then $\Gamma_{2N} = (\Gamma_N)^2$, or $\Gamma_N = (\Gamma_1)^N$.

Let us call "normalized signal entropy" the quantity

$$\begin{aligned} \tilde{H} &= (1/n) \lim_{n\to\infty} \tilde{H}_n \\ \tilde{H}_n &= -\int \cdots \int p_{2n}(\mathbf{x}, \mathbf{y}) \log_e[\Gamma_{2n}\, p_{2n}(\mathbf{x}, \mathbf{y})]\, d^n\mathbf{x} \cdot d^n\mathbf{y} \end{aligned} \tag{6}$$

Let us choose Γ_1 such that $\tilde{H} = 0$ for a real Gaussian (r.v.), i.e.,[7]

$$\Gamma_1 = (2\pi e I)^{-1/2} \tag{7}$$

where $I = \langle x^2 \rangle$. In this paper we compare the normalized entropies for analytic signal amplitudes of generalized Gaussian fields. Because of our choice [Eq. (7)] and the property given in Appendix A, the maximum entropy is zero, so that all the entropies are negative quantities.

We shall successively use the two previous definitions [Eqs. (1) and (6)] to compare the entropy for generalized Gaussian fields.

3. THERMODYNAMIC ENTROPY FOR PSEUDO-GAUSSIAN FIELDS

The calculation of S[Eq. (1)] is very complicated for a nondiagonal density matrix, so that we shall try to reduce our comparison to diagonal matrices. Every quasistationary[2] em field $E(t) = Z(t)\, e^{i\omega_0 t}$ can be made stationary simply by introduction of a uniform phase ϕ. We obtain

$$\tilde{E}(t) = Z(t)\, e^{i\phi + i\omega_0 t} \tag{8}$$

which is fully stationary.[1,2]

Of course if $E(t)$ is a Gaussian r.f., $\tilde{E}(t)$ is no longer Gaussian [unless $E(t)$ is chaotic]; we shall call $\tilde{E}(t)$ a pseudo-Gaussian field. Let us compare the entropy for generalized pseudo-Gaussian fields, using Eqs. (1) and (2); we have

$$S[\tilde{E}(t)] = -k \sum_n \rho_{nn} \log \rho_{nn} \tag{9}$$

[2] Quasistationary means that the moment $\langle E(t_1)E^*(t_2)\rangle$ is a function of $(t_2 - t_1)$ but $\langle E(t_1)E(t_2)\rangle$ is a function of t_1, t_2. All the generalized Gaussian fields except the chaotic one are quasistationary.[1,2]

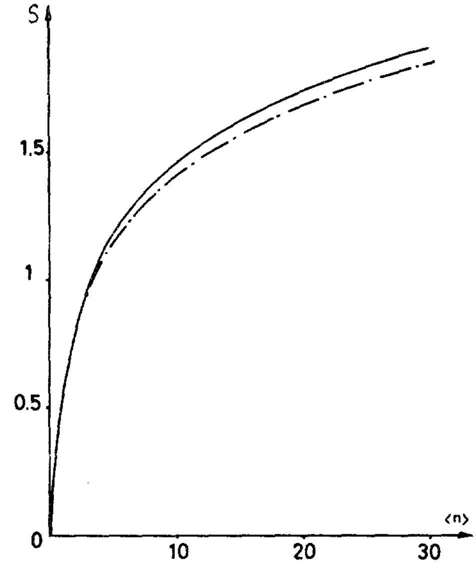

Fig. 1. Thermodynamic entropy for monomode fields as a function of the mean number $\langle n \rangle$ of photons in the mode. Full line: chaotic field $ze^{i\omega_0 t}$. Broken line: real pseudo-Gaussian field $xe^{i(\omega_0 t+\phi)}$.

where the matrix elements ρ_{nn} are formally identical to counting distributions of the corresponding Gaussian fields[3] $E(t)$ since they have the same intensity $I(t) = |\tilde{E}(t)|^2 = |E(t)|^2$.

The expressions of entropy for the chaotic field, the real pseudo-Gaussian field $E(t) = xe^{i(\phi+\omega_0 t)}$, and the generalized pseudo-Gaussian field[1,2] $E(t) = (x + iy)\, e^{i(\phi+\omega_0 t)}$ with[4] $\langle x \cdot y \rangle = \rho\sigma^2$ are as follows:

$$S^{(\mathrm{ch})} = k(\langle n \rangle + 1) \log(\langle n \rangle + 1) - k\langle n \rangle \log\langle n \rangle \tag{10a}$$

$$S^{(\mathrm{real})} = -k\alpha \sum_n \frac{\Gamma(n + \frac{1}{2})}{n!}\, a^n \log \frac{\alpha\Gamma(n + \frac{1}{2})\, a^n}{n!} \tag{10b}$$

[3] See Ref. 1, Eqs. (5.10) and (4.7). There is mistake in Eq. (5.10), which must be replaced by

$$p_n(T) = (1 - \rho^2)^{1/2} \frac{a^n}{(1 + a)^{n+1}} \sum_{p=0}^{\infty} \frac{(n + 2p)!}{n!p!p!} \frac{\rho^{2p}}{(1 + a)^4 p}$$

where $a = 2\sigma^2 T(1 - \rho^2)$, and ρ is the correlation coefficient of x and y.

[4] Notice that $\rho = 0$ corresponds to the monomode chaotic field and $\rho = 1$ to a field which has the same intensity as a real Gaussian field.

with $a = 2\langle n\rangle/(2\langle n\rangle + 1)$ and $\alpha = [\pi(2\langle n\rangle + 1)]^{-1/2}$; and

$$S^{(\text{generalized})} = -k \sum_{n=0}^{\infty} (1-\rho^2)^{1/2} \frac{a^n}{(1+a)^{n+1}} \sum_{p=0}^{\infty} \frac{(n+2p)!}{n!p!p!} \frac{\rho^{2p}}{(1+a)^{4p}}$$

$$\times \log \left\{(1-\rho^2)^{1/2} \frac{a^n}{(1+a)^{n+1}} \sum_{p=0}^{\infty} \frac{(n+2p)!}{n!p!p!} \frac{\rho^{2p}}{(1+a)^{4p}}\right\} \quad (10c)$$

where a has the same meaning as in the previous equation.

These expressions can be computed; they are shown in Fig. 1 (entropy for a chaotic field and a real pseudo-Gaussian field as a function of $\langle n\rangle$) and Fig. 2 [Eqs. (10a)–(10c) for generalized pseudo-Gaussian fields for $\langle n\rangle = 30$, as a function of ρ].

In conclusion, for all the monomode stationary pseudo-Gaussian fields $ze^{i(\omega_0 t+\phi)}$, where z is a complex Gaussian random variable (r.v.) and ϕ a uniform phase, the entropy is situated between two limits: The maximum is the entropy of the chaotic field ($\rho = 0$); the minimum is the entropy of the real pseudo-Gaussian field $xe^{i(\omega_0 t+\phi)}$ (or $\rho = 1$).

The calculation of entropy for multimode Gaussian fields is rather complicated because of the correlation between modes,[2] except for the chaotic field, for which entropy is no more than the sum of the entropies for each mode.

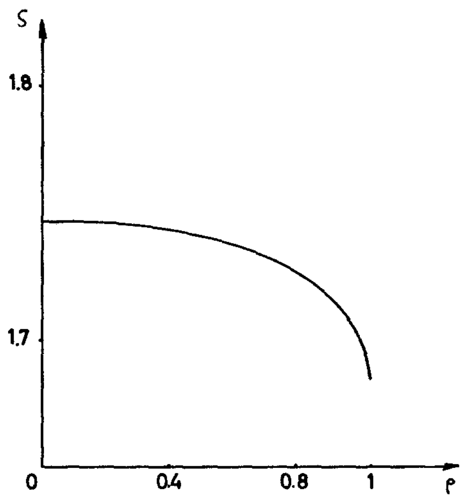

Fig. 2. Thermodynamic entropy for monomode pseudo-Gaussian field $ze^{i(\omega_0 t+\phi)}$ as a function of $\rho = \langle xy\rangle/\sigma_x\sigma_y$ for $\langle n\rangle = 30$.

4. SIGNAL ENTROPY FOR GENERALIZED GAUSSIAN FIELDS

For any Gaussian r.f. $Z(t) = x(t) + iy(t)$, the $2n$-dimensional probability distribution can be written as[5]

$$p(x_1, ..., x_n ; y_1, ..., y_n) \frac{1}{(2\pi)^n |A|^{1/2}} \exp\left(-\frac{1}{2} U^{\dagger} A^{-1} U\right) \tag{11}$$

where U is the $2n$-dimensional vector $(x_1, ..., x_n ; y_1, ..., y_n) = (U_1, ..., U_{2n})$ and A is the covariance matrix whose elements are $A_{ij} = \langle U_i U_j \rangle$; $|A|$ is its determinant.

There exists a unitary transformation S so that the vector U becomes $V = (v_1, ..., v_{2n})$; and the covariance matrix A becomes a diagonal matrix whose diagonal elements are its positive eigenvalues $(\lambda_1, ..., \lambda_{2n})$. On this new basis, Eq. (11) becomes

$$p(v_1, ..., v_{2n}) = \frac{1}{(2\pi)^n (\lambda_1, ..., \lambda_{1n})^{1/2}} \exp\left(-\frac{1}{2} \sum_{i=1}^{n} \frac{v_i^2}{\lambda_i}\right) \tag{12}$$

Thus Shannon's entropy is

$$\begin{aligned} H_n &= -\int \cdots \int p(\mathbf{v}) \log p(\mathbf{v})\, d\mathbf{v} \\ &= +\int \cdots \int p(\mathbf{v}) \left\{ \log[(2\pi)^n |A|^{1/2}] + \sum (v_i^2/\lambda_i) \right\} d\mathbf{v} \end{aligned} \tag{13}$$

which is

$$H_n = \log[(2\pi e)^n |A|^{1/2}] \tag{14}$$

The normalized signal entropy [Eqs. (6) and (7)], related to H_n by the relation

$$\tilde{H}_n = H_n - \log(\Gamma_1)^{2n} \tag{15}$$

is given for any Gaussian field [cf. Eq. (13)] by

$$\tilde{H} = \lim_{n\to\infty} \log[|A|^{1/2n}/I] \tag{16}$$

where A is the $2n \times 2n$ covariance matrix built on $(x_1, ..., x_n ; y_n, ..., y_n)$ and $I = \langle x_i^2 \rangle = \langle y_i^2 \rangle$.

If the variances of x and y are different, $\langle x_i^2 \rangle = I_x$ and $\langle y_i^2 \rangle = I_y$, Eq. (16) is still valid with $I = (I_x I_y)^{1/2}$.

We must notice that the normalized signal entropy [Eq. (16)] is invariable through multiplication of each (x_i, y_i) by the same constant factor.

Now let us compare $\tilde{H}$ for different Gaussian fields.

[5] Mandel and Wolf,[9] Eqs. (4) and (38).

4.1. Real Gaussian Field

The matrix A reduces to an $n \times n$ matrix A_x built on $(x_1, ..., x_n)$ so that Eq. (12) must be replaced by

$$p(v_1, ..., v_n) = \frac{1}{(2\pi)^{n/2} |A_x|^{1/2}} \exp - \left(\frac{1}{2} \sum_{i=1}^{n} \frac{v_i^2}{\lambda_i}\right) \tag{17}$$

Shannon's entropy is then[7]

$$H_n^{(\text{real}} = \log[(2\pi e)^{n/2} |A_x|^{1/2}] \tag{18}$$

and the normalized signal entropy is

$$\tilde{H}^{(\text{real})} = \lim_{n\to\infty} \tfrac{1}{2} \log(|A_x|^{1/n}/I_x) \tag{19}$$

where $I_x = \langle x_i^2 \rangle$.

Notice that $\tilde{H}$ in Eq. (19) is still invariant through the transformation $I_x = \langle x_i^2 \rangle \to \alpha I_x$, since $|\alpha A_x| = \alpha^n |A_x|$. Moreover, this quantity is negative since $|A_x|^{1/n} < I_x$ for any positive-definite matrix A_x. The previous inequality holds in a strict sense because A_x is not a diagonal matrix when $X(t)$ has correlations, which is the case for any quasimonochromatic field we consider here.

Let us call $\mathcal{O}\!\!\ell_x = (1/I_x) A_x$ the "normalized" correlation matrix such that $\langle x_i^2 \rangle = 1$. The entropy [Eq. (19)] is then

$$\tilde{H}^{(\text{real})} = \lim_{n\to\infty} \log |a_x|^{1/2n} \tag{20}$$

4.2. Independent Gaussian Field [$x(t)$ and $y(t)$ are independent Gaussian r.f.'s]

When $x(t)$ and $y(t)$ are independent r.f.'s, the distribution in Eq. (11) factorizes, and entropy is simply $H(x) + H(y)$, where $H(x)$ [resp. $H(y)$] correspond to $x(t)$ [resp. $y(t)$] (see Section 2.2 and Appendix A); we thus have

$$H_n^{\text{indep}} = H_n(x) + H_n(y) = \log[(2\pi e)^n |A_x A_y|^{1/2}] \tag{21}$$

$$\tilde{H}^{\text{indep}} = \lim_{n\to\infty} \log[|A_x A_y|^{1/2n}/(I_x I_y)^{1/2}] \tag{22}$$

where A_x and A_y are the correlation matrices of $x(t)$ and $y(t)$. When $X(t)$ and $Y(t)$ have the same distribution, Eq. (22) can be simplified and we obtain

$$\tilde{H}^{\text{indep}} = 2\tilde{H}^{\text{real}} = \lim_{n\to\infty} \log |\mathcal{O}\!\!\ell_x|^{1/n} \tag{23}$$

where $\mathcal{O}\!\!\ell_x = A_x/I_x$ as previously.

Notice that in Section 4.1 and here we compared two fields with different mean intensity [since for the real Gaussian field $\langle I(t)\rangle = I_x = \langle x^2(t)\rangle$, while for the independent one $\langle I(t)\rangle = I_x + I_y$]. But Eqs. (20) and (22) are both invariant by the transformation $I_x \to \alpha I_x$, so that Eq. (23) is valid whatever the mean intensity. In Eq. (23) we have $|\mathcal{O}_x| < 1$, so that the normalized entropies of independent Gaussian fields are negative, too.

The main conclusion of this section is that *the real Gaussian field has a normalized entropy greater than the independent Gaussian field.*

This result seems to be contradictory with Appendix A, which states that the independent Gaussian field has a maximum Shannon entropy. As a matter of fact, Shannon compares two r.f.'s with the same *a priori* distributions $p(x)$ and $q(y)$.

For our problem, with Shannon's definition, the signal entropy is given by

$$H^{\text{indep}} = \lim_{n\to\infty} \log[(2\pi e)| A_x |^{1/n}]$$
$$H^{\text{real}} = \lim_{n\to\infty} \log[(2\sqrt{\pi e})| A_x |^{1/2n}] \tag{24}$$

So that

$$H^{\text{indep}} - H^{\text{real}} = \tfrac{1}{2} \log[\pi e \,| A_x |^{1/n}].$$

This quantity can be positive as well as negative, according to the choice of units.

The normalization [Eqs. (6) and (7)] only introduces the constant Γ_{2n} which depends on the r.f. $Z(t)$. This constant is absolutely necessary when we want to compare entropies for a real r.f. $X(t)$ and a complex r.f. $Z(t)$.

4.3. Correlated Gaussian Field [$x(t)$ and $y(t)$ are correlated r.f.'s]

A "correlated" Gaussian field has a matrix A_{xy} which is not zero. Shannon's entropy for such a signal is

$$H_n^{\text{correl}} = \log[(2\pi e)^n | A^{\text{correl}} |^{1/2}] \tag{25}$$

where $|A^{\text{correl}}|$ is the determinant of the $2n \times 2n$ covariant matrix. The normalized entropy for such a field is

$$\tilde{H}^{\text{correl}} = \lim_{n\to\infty} \log(| \mathcal{O}^{\text{correl}} |^{1/2n}) \tag{26}$$

where $\mathcal{O}^{\text{correl}} = (1/I_x)\, A^{\text{correl}}$ as previously.

We can compare now the entropy for a correlated Gaussian field and an independent Gaussian field, using the result of Appendix A. We suppose they have the same determinants $|\mathcal{O}_x|$ and $|\mathcal{O}_y|$; we consider a real Gaussian field of determinant $|\mathcal{O}_x|$. We have

$$\tilde{H}^{\text{correl}} < \tilde{H}^{\text{indep}} < \tilde{H}^{\text{real}} \tag{27}$$

So that[6]

$$|\mathcal{A}|^{\text{correl}} < |\mathcal{A}_x|^2 < |\mathcal{A}_x| \qquad (28)$$

Of course, the chaotic field is a particular case of the correlated Gaussian fields, so that the conclusions of Section 4 are clearly distinct from those of Section 3.

5. CONCLUSION AND DISCUSSION

It is well known that the chaotic field has maximum thermodynamic entropy defined as $\mathrm{Tr}(\rho \log \rho)$. We have computed the values of entropy for a monomode generalized pseudo-Gaussian field $ze^{i(\omega_0 t+\phi)}$, where z is a Gaussian r.v. and ϕ is a uniform phase. We proved that the minimum entropy concerns the real Gaussian field $xe^{i(\omega_0 t+\phi)}$.

The signal entropy defined by Shannon necessitates a slight modification to be adequate for the study of em fields, i.e., we introduce a normalization additive constant $\log \Gamma_n$ so that we can compare signal entropies for real and complex signals. The result is very different from the thermodynamic entropy, since the real Gaussian field has maximum normalized signal entropy. We study independent Gaussian fields [$x(t)$ and $y(t)$ independent r.f.'s) and correlated Gaussian fields [$x(t)$ and $y(t)$ correlated r.f.'s as is the case for the chaotic field] and we prove that the chaotic field has a smaller entropy than the independent one.

We must notice that the signal entropy is not adapted to the study of monochromatic fields. In fact, for any monochromatic field the correlation normalized matrix ($\langle x^2\rangle = \langle y^2\rangle = 1$) is

$$A = \left(\begin{array}{c|c} (1) & \rho \times (1) \\ \hline \rho \times (1) & (1) \end{array}\right)$$

where (1) is the $n \times n$ matrix all of whose elements are unity, and $\rho = \langle xy\rangle$. From Eq. (26) we see that the signal entropy of a monomode Gaussian field is

$$\tilde{H}^{\text{monomode}} = \lim_{n\to\infty} \log[(1-\rho^2)^{1/2}|(1)|^{1/n}]$$

But the determinant of the matrix (1) is null, so that the signal entropy diverges for a monochromatic field.

[6] Let us denote by B the $n \times n$ covariance matrix built on the complex r.v. $(z_1 \cdots, z_n)$. We can use Eq. (28) to prove the following relation between the determinants of a *chaotic* field (see Appendix B): $|B|^2 = |A| < |A_x|^2$.

APPENDIX A

Let us set

$$H(\mathbf{x}, \mathbf{y}) = -\int p(\mathbf{x}, \mathbf{y}) \log p(\mathbf{x}, \mathbf{y})\, d\mathbf{x}\, d\mathbf{y} \tag{A.1}$$

where $\mathbf{x} = (x_1, ..., x_n)$ and $\mathbf{y} = (y_1, ..., y_n)$ are the values of $X(t)$ and $Y(t)$ at times $t_1, ..., t_n$.

$H(\mathbf{x}, \mathbf{y})$ can also be written, using *a priori* and conditional distributions,

$$\begin{aligned} H(\mathbf{x}, \mathbf{y}) &= -\int p(\mathbf{x}, \mathbf{y}) \log \left[p(\mathbf{x})\, q(\mathbf{y}) \frac{p(\mathbf{x}, \mathbf{y})}{p(\mathbf{x})\, q(\mathbf{y})} \right] d\mathbf{x}\, d\mathbf{y} \\ &= H(\mathbf{x}) + H(\mathbf{y}) - I(\mathbf{x}, \mathbf{y}) \end{aligned} \tag{A.2}$$

with

$$I(\mathbf{x}, \mathbf{y}) = \int p(\mathbf{x}, \mathbf{y}) \log \frac{p(\mathbf{x})\, q(\mathbf{y})}{p(\mathbf{x}, \mathbf{y})}\, d\mathbf{x}\, d\mathbf{y} \tag{A.3}$$

using the relation $\ln a \leqslant a - 1$, we have

$$I(\mathbf{x}, \mathbf{y}) \leqslant 0 \qquad \forall \mathbf{x}, \mathbf{y} \tag{A.4}$$

In Eq. (A.4) the equality corresponds to the independence of $\mathbf{x}$ and $\mathbf{y}$. Thus

$$H(\mathbf{x}, \mathbf{y}) \leqslant H(\mathbf{x}) + H(\mathbf{y}) \tag{A.5}$$

The entropy of a complex r.f. less than the sum of the entropies of its real and imaginary parts.

Shannon established that $H(\mathbf{x})$ is maximum for a real Gaussian r.f. $X(t)$; thus *the entropy $H(\mathbf{x}, \mathbf{y})$ is maximum if $X(t)$ and $Y(t)$ are two real, independent Gaussian r.f.'s.*

APPENDIX B

The probability distribution for a chaotic field can be written as Eq. (11) or

$$p(z_1, ..., z_n, z_1^*, ..., z_n^*) = [1/(2\pi)^n |B|] \exp(-\tfrac{1}{2} z^\dagger B^{-1} z)$$

Shannon's entropy is thus

$$H_n^{(\mathrm{ch})} = \log[(2\pi e)^n |B|]$$

comparing with Eq. (25), we have

$$|B|^2 = |A|$$

with $B = 2A_x + 2iA_{xy}$.

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